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ON AMICABLE NUMBERS AND THEIR GENERALIZATIONS.¹

By THOMAS E. MASON, Purdue University.

Introduction. Amicable numbers m and n are defined by the equations

$$S(m) = S(n) = m + n,$$

where $S(m)$ means the sum of all the divisors of m , including m and unity. The great interest in amicable numbers is shown by the number of papers concerning them reviewed in Dickson's *History of the Theory of Numbers*. There are nearly as many papers as there are known amicable number pairs. This interest began in very early times when the first pair—220 and 284—was discovered. The belief in their power to bring friendship lasted through a long period. Euler made the most extensive contributions to the methods of discovering such number pairs. The above pair is referred to by Iamblichus² (283–330 A.D.), the second known pair was discovered by Fermat, the third by Descartes. Euler with new methods added fifty-nine others, Legendre one, Paganini one, Seelhoff two, Dickson two. This paper adds fourteen.

Dickson³ defined an amicable k -tuple as k numbers satisfying the equations

$$S(n_1) = S(n_2) = \dots = S(n_k) = n_1 + n_2 + \dots + n_k.$$

He searched for amicable triples only. This paper gives amicable k -tuples for $k = 3, 4, 5, 6$.

Carmichael⁴ defined a multiply amicable number pair as numbers m and n satisfying the equations

$$S(m) = S(n) = t(m + n),$$

where t is an integer. The same generalization applies to the k -tuples. A large number of such multiply amicable number pairs for $t = 2, 3$ and amicable triples for $t = 2$ are readily found by methods indicated in § 2.

No attempt has been made here to find k -tuples in which two or more of the numbers are alike. That such exist is shown in some of the sets of triples found by Dickson. Any multiply perfect number of multiplicity k (equals 2 for ordinary perfect numbers) is, of course, a multiply amicable k -tuple where all the k numbers are alike.

1. Amicable Number Sets. One of Euler's methods is to search for amicable numbers of the form apq and arf , where p, q, r are primes and a and f are composite, but prime to each other and to p, q , and r . Let $f = (g - 1)(h - 1)$, where $g - 1$ and $h - 1$ are primes. Then $S(f) = gh$. Put $p + 1 = hx$,

¹ Read before the American Mathematical Society, Chicago, December, 1920.

² See Dickson, *History of the Theory of Numbers*, vol. 1, 1919, pp. 38–50.

³ In this MONTHLY, 1913, 84–92.

⁴ In this MONTHLY, 1919, 399.

$r + 1 = xy$ and $q + 1 = gy$. We shall have

$$ghxy S(a) = S(af r) = a(pq + fr) = a\{(hx - 1)(gy - 1) + f(xy - 1)\}.$$

In the equation formed by setting the first member above equal to the fourth, replace $a/(2a - S(a))$ by b/c and $bf - bgh + cgh$ by e and the result can be written

$$(E) \quad (ex - bg)(ey - bh) = b^2gh + be(f - 1).$$

In applying this method a and f are assumed. $S(f)$ gives g and h . It remains to solve for x and y , and hence p , q and r . The restriction that g and h of formula (E) be the g and h of $f = (g - 1)(h - 1)$ is not necessary. It is sufficient to take g and h of formula (E) any factorization of $S(f)$. This more general g and h will occasionally give amicable number pairs when the restricted factorization of Euler fails to do so. The methods used to find the amicable number pairs of this paper were those of Euler with the modification just mentioned and the change in the fifth method of Euler made by Dickson as noted below.

If we seek for an amicable k -tuple $an_1, an_2, \dots, an_k$ where

$$S(n_1) = \dots = S(n_k)$$

we have

$$S(a)S(n_1) = a(n_1 + n_2 + \dots + n_k) \text{ or } S(a)/a = (n_1 + n_2 + \dots + n_k)/S(n_1).$$

In applying this method $n_1, n_2, \dots, n_k$ are chosen and a value of a is sought. This method is given by Dickson in his search for amicable triples. It has been used in this paper more extensively than any other method. It is a modification of Euler's fifth method.

The use of the methods mentioned has produced the following:

Amicable pairs.

$$\begin{array}{lll} 2 \cdot 5^2 \cdot 31 \left\{ \begin{array}{l} 47 \cdot 149 \\ 19 \cdot 359 \end{array} \right. & 2^3 \left\{ \begin{array}{l} 13 \cdot 23 \cdot 251 \\ 97 \cdot 863 \end{array} \right. & 3^2 \cdot 7^2 \cdot 13 \cdot 19 \left\{ \begin{array}{l} 11 \cdot 10499 \\ 89 \cdot 1399 \end{array} \right. \\ 2^3 \left\{ \begin{array}{l} 11 \cdot 41 \cdot 173 \\ 71 \cdot 1217 \end{array} \right. & 2^3 \left\{ \begin{array}{l} 13 \cdot 23 \cdot 1109 \\ 71 \cdot 5179 \end{array} \right. & 3^2 \cdot 7^2 \cdot 13 \cdot 19 \left\{ \begin{array}{l} 17 \cdot 23 \cdot 1335949 \\ 3079 \cdot 187379 \end{array} \right. \\ 2^3 \left\{ \begin{array}{l} 11 \cdot 31 \cdot 233 \\ 127 \cdot 701 \end{array} \right. & 2^3 \left\{ \begin{array}{l} 17 \cdot 19 \cdot 281 \\ 53 \cdot 1879 \end{array} \right. & 3^3 \cdot 5^3 \left\{ \begin{array}{l} 29 \cdot 41 \cdot 43 \cdot 59 \\ 19 \cdot 131 \cdot 1259 \end{array} \right. \\ 2^3 \left\{ \begin{array}{l} 11 \cdot 31 \cdot 2099 \\ 79 \cdot 10079 \end{array} \right. & 3^2 \cdot 5^2 \cdot 31 \left\{ \begin{array}{l} 29 \cdot 41 \cdot 43 \cdot 59 \\ 19 \cdot 131 \cdot 1259 \end{array} \right. & 3^4 \cdot 5 \cdot 11 \left\{ \begin{array}{l} 41 \cdot 599 \\ 59 \cdot 419 \end{array} \right. \\ 2^3 \left\{ \begin{array}{l} 13 \cdot 23 \cdot 149 \\ 199 \cdot 251 \end{array} \right. & & 3^4 \cdot 5 \cdot 11 \cdot 59 \left\{ \begin{array}{l} 89 \cdot 5309 \\ 477899 \end{array} \right. \end{array}$$

Amicable triples.

$3 \cdot 89 \cdot a, 11 \cdot 29 \cdot a, 359 \cdot a$ ($a = 2^8 \cdot 5 \cdot 19 \cdot 37 \cdot 73$) and ($a = 2^{20} \cdot 7 \cdot 13^2 \cdot 31 \cdot 61 \cdot 127 \cdot 337$)
 $17 \cdot 79 \cdot b, 19 \cdot 71 \cdot b, 1439 \cdot b$ ($b = 2^{15} \cdot 5^2 \cdot 11 \cdot 31 \cdot 43 \cdot 257$)
 $3 \cdot 17 \cdot 109 \cdot c, 29 \cdot 263 \cdot c, 71 \cdot 109 \cdot c$ ($c = 2^9 \cdot 7 \cdot 11^2 \cdot 19$) and ($c = 2^{14} \cdot 7 \cdot 11 \cdot 19 \cdot 151$).

Amicable quadruples.

$7 \cdot 107 \cdot a$, $11 \cdot 71 \cdot a$, $17 \cdot 47 \cdot a$, $863 \cdot a$ ($a = 2^8 \cdot 3^3 \cdot 5 \cdot 37 \cdot 73$)
 $17 \cdot 79 \cdot b$, $19 \cdot 71 \cdot b$, $23 \cdot 59 \cdot b$, $1439 \cdot b$ ($b = 2^5 \cdot 3^5 \cdot 5 \cdot 13$)
 $19 \cdot 71 \cdot c$, $23 \cdot 59 \cdot c$, $29 \cdot 47 \cdot c$, $1439 \cdot c$ ($c = 2^{15} \cdot 3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 43 \cdot 257$)
 $5 \cdot 139 \cdot d$, $13 \cdot 59 \cdot d$, $19 \cdot 41 \cdot d$, $839 \cdot d$, ($d = 2^{33} \cdot 3^6 \cdot 7 \cdot 23 \cdot 83 \cdot 137 \cdot 331 \cdot 547 \cdot 1093$
 $\cdot 43691 \cdot 131071$),
 $(d = 2^{33} \cdot 3^{10} \cdot 7 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 131071)$,
 $(d = 2^{37} \cdot 3^6 \cdot 7 \cdot 23 \cdot 83 \cdot 137 \cdot 331 \cdot 547 \cdot 1093 \cdot 43691 \cdot 174763 \cdot 524287)$ and
 $(d = 2^{37} \cdot 3^{10} \cdot 7 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 174763 \cdot 524287)$.
 $17 \cdot 1999 \cdot e$, $23 \cdot 1499 \cdot e$, $71 \cdot 499 \cdot e$, $149 \cdot 239 \cdot e$, ($e = 2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13$), ($e = 2^9 \cdot 3^3 \cdot 5 \cdot 11$),
 $(e = 2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 151)$, ($e = 2^{14} \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 127 \cdot 151$),
 $(e = 2^{25} \cdot 3^3 \cdot 5^2 \cdot 19 \cdot 683 \cdot 2731 \cdot 8191)$, ($e = 2^{33} \cdot 3^4 \cdot 7 \cdot 11^3 \cdot 61 \cdot 83 \cdot 331 \cdot 43691 \cdot 131071$)
 and ($e = 2^{37} \cdot 3^4 \cdot 7 \cdot 11^3 \cdot 61 \cdot 83 \cdot 331 \cdot 43691 \cdot 174763 \cdot 524287$).

Amicable quintuples.

$17 \cdot 79 \cdot a$, $19 \cdot 71 \cdot a$, $23 \cdot 59 \cdot a$, $29 \cdot 47 \cdot a$, $1439 \cdot a$ ($a = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 43 \cdot 257$).
 $17 \cdot 439 \cdot b$, $29 \cdot 263 \cdot b$, $43 \cdot 179 \cdot b$, $59 \cdot 131 \cdot b$, $71 \cdot 109 \cdot b$,
 $(b = 2^{33} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 83 \cdot 331 \cdot 43691 \cdot 131071)$ and
 $(b = 2^{37} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 83 \cdot 331 \cdot 43691 \cdot 174763 \cdot 524287)$.

Amicable sextuples.

$17 \cdot 1999 \cdot a$, $23 \cdot 1499 \cdot a$, $59 \cdot 599 \cdot a$, $71 \cdot 499 \cdot a$, $79 \cdot 449 \cdot a$, $35999 \cdot a$
 $(a = 2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^5 \cdot 11^2 \cdot 13 \cdot 19^2 \cdot 31^2 \cdot 61 \cdot 83 \cdot 127 \cdot 223 \cdot 331 \cdot 7019 \cdot 112303 \cdot 898423$
 $\cdot 616318177)$.

$23 \cdot 1499 \cdot b$, $59 \cdot 599 \cdot b$, $71 \cdot 499 \cdot b$, $79 \cdot 449 \cdot b$, $149 \cdot 239 \cdot b$, $179 \cdot 199 \cdot b$, where b is to be chosen as follows: take any multiply perfect number of multiplicity 6 which contains the factors $79 \cdot 157$ and which does not contain any of the other factors multiplying b in the numbers above, from this number omit $79 \cdot 157$ and what remains will give a value for b . The known multiply perfect numbers furnish at least six such values of b .

2. Multiply Amicable Number Sets. If we seek for multiply amicable number pairs of the form apq and arf , where p , q and r are primes and a and f are composite, but prime to each other and to p , q and r , we have the equations

$$S(a)S(pq) = S(a)S(rf) = ta(pq + rf).$$

By the same process as that used in § 1 we arrive at equation (E), using the same substitutions, except that we replace $ta/(2ta - S(a))$ by b/c . The numbers p , q , and r depend only on the ratio b/c . It is evident, therefore, that the same numbers p , q , r and f that will form an amicable number pair with a given a will serve to form a multiply amicable number pair if we can find a new a' such that $ta'/(2ta' - S(a')) = a/(2a - S(a)) = b/c$. For example, the numbers $a \cdot 11 \cdot 41 \cdot 173$

By use of this table in connection with a table¹ of the known multiply perfect numbers about six hundred multiply amicable number pairs can be obtained of multiplicity $t = 2, 3$.

Amicable number sets of the form am and an can be found by the use of the equations $S(m) = S(n)$ and $S(a)S(m) = ta(m + n)$. From the latter equation we have $S(a)/ta = (m + n)/S(m)$. Select m and n so that $S(m) = S(n)$ and then find the ratio $(m + n)/S(m)$ and seek a value for a . This method will serve for amicable k -tuples.

For the case of multiplicity 2 we have the following²: If m and n are amicable numbers and a is a perfect number, prime to m and n , then am and an are multiply amicable numbers of multiplicity 2. By hypothesis $S(a) = 2a$ and $S(m) = S(n) = m + n$, and hence, $S(a)S(m) = S(a)S(n) = 2a(m + n)$.

A few multiply amicable number sets are given here that were not found by means of the table above.

For $t = 2$:

$$53 \cdot 431 \cdot a, 23327 \cdot a \quad (a = 2^8 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 37 \cdot 73)$$

$13 \cdot 23 \cdot 1109 \cdot b, 71 \cdot 5179 \cdot b$ ($b = 2^{15} \cdot 3^4 \cdot 11^3 \cdot 17 \cdot 31 \cdot 43 \cdot 61 \cdot 257$) and for the same value of b

$$13 \cdot 23 \cdot 251 \cdot b, 97 \cdot 863 \cdot b \text{ and } 13 \cdot 23 \cdot 149 \cdot b, 199 \cdot 251 \cdot b$$

$$7 \cdot 59 \cdot c, 19 \cdot 23 \cdot c \quad (c = 2^7 \cdot 3^3 \cdot 5)$$

$$7 \cdot 53 \cdot d, 17 \cdot 23 \cdot d \quad (d = 2^6 \cdot 3^3 \cdot 5)$$

$$43 \cdot 167 \cdot 449 \cdot e, 59 \cdot 131 \cdot 419 \cdot e \quad (e = 2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot 31)$$

$$7 \cdot 89 \cdot f, 23 \cdot 29 \cdot f \quad (f = 2^{13} \cdot 3^3 \cdot 5 \cdot 127)$$

$$59 \cdot 599 \cdot g, 71 \cdot 499 \cdot g \quad (g = 2^{20} \cdot 3 \cdot 5^2 \cdot 7 \cdot 31 \cdot 127).$$

A multiply amicable triple with $t = 2$ is given by $59 \cdot 331 \cdot a, 23 \cdot 829 \cdot a, 19919 \cdot a$, where a is found by omitting the factor 47 from a multiply perfect number of multiplicity 6 and using the remainder of that number as a , provided that the multiply perfect number does not contain any of the factors 23, 59, 331, 829, 19919.

Conclusion. The methods of finding amicable number sets are very largely those of trial. Experience in working with such numbers will suggest the likely numbers to try, but there is no sure guide yet known. The number of cases that need to be tried out becomes very large for some of the larger numbers that we might choose for $n_1, n_2, \dots, n_k$ in seeking k -tuples of the form $an_1, an_2, \dots, an_k$. For example, the number 1108800 is the sum of the divisors of at least twenty different sets of prime factors no one of which is less than 19. If we are using the method suggested by Dickson and are seeking amicable sextuples we shall have as many possibilities as there are combinations of twenty, six at a time. The same set offers ${}_{20}C_2$ possibilities for amicable pairs, ${}_{20}C_3$ for triples, etc. Any systematic search for amicable numbers among the large numbers will furnish a vast amount of work.

The author of this paper was attracted to the subject on reading the section

¹ Carmichael and Mason, *Proceedings of the Indiana Academy of Science*, 1911, pp. 257-270.

² Indicated but not explicitly stated by Carmichael, in this MONTHLY, 1919, 399.

on amicable numbers in Dickson's *History of the Theory of Numbers* and seeing how large a number of papers had been written on the subject and how relatively few numbers had been found. After some time spent in searching he is convinced that any one with a little skill in manipulation of numbers, considerable patience, and access to the Lehmer's *List of Prime Numbers*, can add to the known list of amicable number pairs, amicable k -tuples or multiply amicable number sets.

AN ELEMENTARY TREATMENT OF FOURIER'S SERIES.

By GEORGE D. BIRKHOFF,¹ Harvard University.

The aim of this note is to treat the "remainder" after $n + 1$ terms of the Fourier's series for a given function $f(x)$:

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots,$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx dx.$$

It will be assumed that the function $f(x)$ is periodic of period 2π and continuous together with its first three derivatives.

Let $T_n(x)$ stand for the sum of the first $n + 1$ terms of the above series; its derivative $T'_n(x)$ is seen at once not to exceed in numerical value

$$(|a_1| + |b_1|) + 2(|a_2| + |b_2|) + \cdots + n(|a_n| + |b_n|).$$

Furthermore, by integration by parts and use of the periodicity of $f(x)$, we find

$$\begin{aligned} a_k &= -\frac{1}{\pi k} \int_{-\pi}^{+\pi} f'(x) \sin kx dx = -\frac{1}{\pi k^2} \int_{-\pi}^{+\pi} f''(x) \cos kx dx \\ &= \frac{1}{\pi k^3} \int_{-\pi}^{+\pi} f'''(x) \sin kx dx. \end{aligned}$$

Consequently $|a_k|$, and similarly $|b_k|$, is less than $2F_3/k^3$ where F_3 is the maximum of $|f'''(x)|$. We conclude that $|T'_n(x)|$ is not greater than

$$4F_3 \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) < 4F_3 \left(1 + \left(1 - \frac{1}{2} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \right),$$

¹ Professor Birkhoff's earlier paper in this MONTHLY, "Note on certain quadratic number systems for which factorization is unique," appeared almost exactly fifteen years ago. His first published paper, in collaboration with H. S. Vandiver, "On the integral divisors of $a^n - b^n$," was published in *Annals of Mathematics*, 1904. His second paper appeared also in *Annals* . . . , 1905. His third paper was a thirty page memoir in *Transactions of the American Mathematical Society*, 1906; and his fourth paper is the one referred to above. A complete collection of his mathematical papers 1904-1919, 37 in number, is preserved in a bound volume at the mathematical seminary of Brown University.—EDITOR.